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MATHEMATICS
TARGET IIT JEE

DEFINITE INTEGRATION

THEORY AND EXERCISE BOOKLET

CONTENTS

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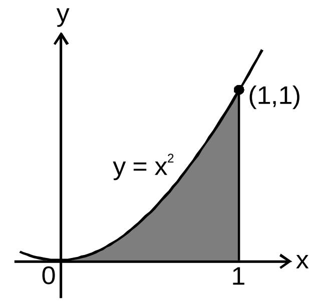
JEE Syllabus :

Definite integrals and their properties, application of definite integrals to the determination of areas involving simple curves

A. THE AREA PROBLEM



Figure

Use rectangles to estimate the area under the parabola $y = x^2$ from, 0 to 1 Wr first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four

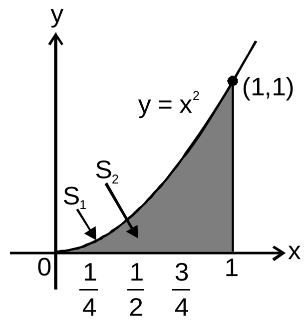
strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$,

$x = \frac{1}{2}$ and $x = \frac{3}{4}$ as in Figure (a).

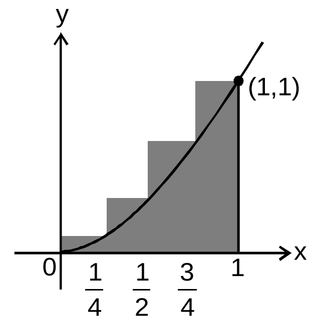
We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure (b)]. In other words, the heights of these rectangle are the values of the function $f(x) = x^2$ at the right end

points of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$,

$\left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.



(a)



(b)

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4},\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}.1^2 = \frac{15}{32} = 0.46875$$

From the Figure (b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in Figure (b) we could use the smaller rectangles in Figure (c) whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its heights is 0). The sum of the areas of these approximating rectangles is

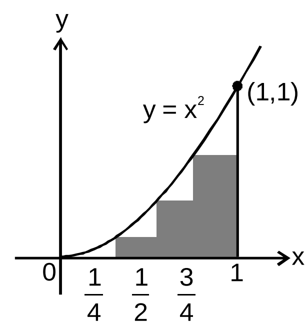


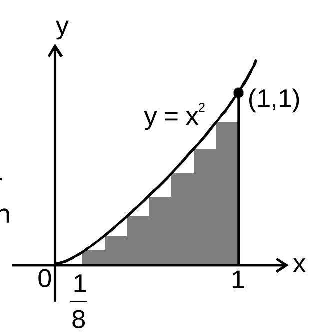
Figure (c)

$$L_4 = \frac{1}{4}.0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 \frac{1}{4} + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

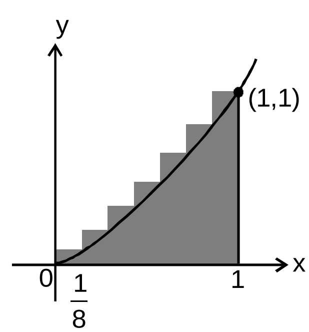
We see that the area of S is larger than L_4, so we have lower and upper estimates for A

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure (d), (e) shows what happens when we divide the region S into eight strips of equal width.



(d) Using left endpoint



(e) Using right endpoint

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A : $0.2734375 < A < 0.3984375$

So one possible answer the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375. We coluld obtain better estimates by increasing the number of strips.

B. PROPERTIES OF DEFINITE INTEGRAL

P-1 : CHANGE OF VARIABLE : The definite integral $\int_a^b f(x)\,dx$ is a number, it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral ;

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr.$$

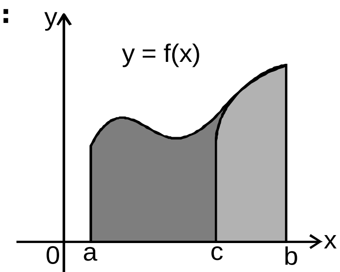
P-2 : CHANGE OF LIMIT : When we defined the definite integral $\int_a^b f(x)dx$, we implicity assumed that a < b. But the definition as a limit of sum makes sense even if a > b. Notice that if we reverse a and b, then Δx changes from (b – a)/n to (a – b)/n.

Therefore $\int_0^a f(x)dx = -\int_a^b f(x)dx$

P-3 : ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION :

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This is not easy to prove in general, but for the case where $f(x) \geq 0$ and a < c < b Property 7 can be seen from the geometric interpretation in Figure : The area under y = f(x) from a to c plus the area from c to b is equal to the total area from a to b.



Figure

Ex.1 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_0^{10} f(x)\,dx$.

Sol. We have $\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$, So $\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$

Ex.2 Find $\int_1^3 (3 - 2x + x^2)\,dx$.

Sol. The function f defined by $f(x) = 3 - 2x + x^2$ is continuous and has antiderivative g defined by

$g(x) = 3x - x^2 + \frac{1}{3}x^3$. Therefore, by the fundamental theorem of the calculus,

$$\int_1^3 (3 - 2x + x^2)\,dx = g(3) - g(1) = (9 - 9 + 9) - (3 - 1 + \frac{1}{3}) = \frac{20}{3}.$$

Ex.3 Evaluate $\int_0^2 |2x - 1|\,dx$.

Sol. We can rewirte the integrand as follows $|2x - 1| = \begin{cases} -(2x-1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$

From this, you can rewrite the integral in two parts.

$$\int_0^2 |2x - 1|\,dx = \int_0^{1/2} (2x-1)\,dx + \int_{1/2}^2 (2x-1)\,dx = [-x^2 + x]_0^{1/2} + [x^2 - x]_{1/2}^2 = \frac{5}{2}$$

Ex.4 Evaluate $\int_{\pi/4}^{\pi/2}\left[\sin x+\left[\frac{2x}{\pi}\right]\right]dx$, (where [∗] denotes greatest integer function)

Sol. $I = \int_{\pi/4}^{\pi/2}\left[\sin x+\left[\frac{2x}{\pi}\right]\right]dx$. Also $\frac{\pi}{4} < x < \frac{\pi}{2} \Rightarrow \frac{1}{2} < \frac{2x}{\pi} < 1 \Rightarrow \left[\frac{2x}{\pi}\right] = 0$ so that $I = \int_{\pi/4}^{\pi/2}[\sin x]dx = 0$.

Ex.5 Find the range of the function $f(x) = \int_0^x |t-1|dt$, where $0 \le x \le 2$.

Sol. Given that $f(x) = \int_0^x |t-1|dt \Rightarrow f(x) = \begin{cases} \int_0^x (1-t)\,dt, & 0 \le x \le 1 \\ \int_0^1 (1-t)\,dt + \int_1^x (t-1)\,dt & 1 < x \le 2 \end{cases} = \begin{cases} x-\frac{x^2}{2}, & 0 \le x \le 1 \\ \frac{x^2}{2}-x+1, & 1 < x \le 2 \end{cases}$

$\Rightarrow$ The range of the function f(x) is [0, 1].

P-4 : $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function i.e. } f(x) = -f(-x) \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function i.e. } f(x) = f(-x) \end{cases}$

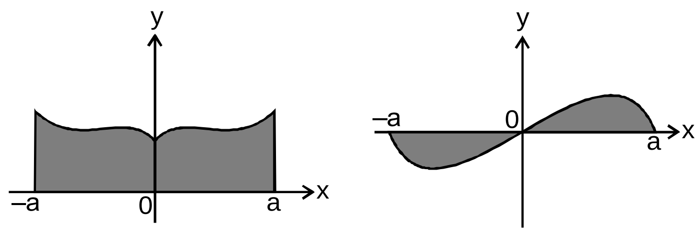
Proof : $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$

In the first integral on the far right side we make the substitution u = –x. Then du = –dx and when x = –a, u = a. Therefore $-\int_0^a f(x)\,dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\,du \Rightarrow \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$

Theorem is illustrated by Figure(a, b) For the case where f is positive and even, part(a) says that the area under y= f(x) from –a to a is twice the area from 0 to a because of symmetry. Thus, part (b) says the integral is 0 because the areas cancel.



Since f(x) = x^6 + 1 satisfies f(–x) = f(x), it is even and so

$$\int_{-2}^{2}(x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx \quad = \left[\frac{1}{7}x^7+x\right]_0^2 = 2\left(\frac{128}{7}+2\right) = \frac{284}{7}$$

Since f(x) = (tan x)/(1 + x^2 + x^4) satisfies f(–x) = –f(x), it is odd and so $\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}\,dx = 0$

P-5 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P-6 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$ **if $f(2a-x) = f(x)$**

$= 0$ **if $f(2a-x) = -f(x)$**

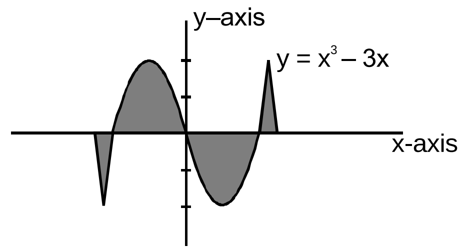
P-7 : $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ **; where 'a' is the period of the function i.e. $f(a+x) = f(x)$**

P-8 : $\int_{a+nT}^{b+aT} f(x)\,dx = \int_a^b f(x)\,dx$ **where f(x) is periodic with period T & $n \in I$**

Remark : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Ex.6 Evaluate $\int_{-2}^{2}(x^3 - 3x)\,dx$.

Sol. The integrand, $f(x) = x^3 - 3x$, is an an odd function; i.e., the equation $f(-x)$ is satisifed for every x. Its graph, drwan in Figure, is therefore symmetric under reflection first about the x-axis and then about the y-axis. If follows that the region above the x-axis has the same areas as the region



below it. We conclude that $\int_{-2}^{2}(x^3 - 3x)\,dx = 0$.

Ex.7 Evaluate $\int_{-n}^{n}(-1)^{[x]}\,dx$, $n \in N$, where [x] denotes the greatest integer function less than or equal to x.

Sol. Let $I = \int_{-n}^{n}(-1)^{[x]}\,dx$. Suppose $f(x) = (-1)^{[x]}$

$\therefore\ f(-x) = (-1)^{[-x]} = (-1)^{-1-[x]},\ x \notin I\ = -(-1)^{-[x]}$

$= -\dfrac{1}{(-1)^{[x]}} = -\dfrac{(-1)^{[x]}}{(-1)^{2[x]}} = -(-1)^{[x]} = -f(x)$, f(x) is odd function.

$\therefore\ I = \int_{-n}^{n}(-1)^{[x]}\,dx = 0$

Ex.8 Show that $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}} dx = \pi^2$.

Sol. Let I = $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}} = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)} dx$ (By prop.) ....(1)

$= \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{(\sin^{2n}x+\cos^{2n}x)} dx$ ....(2)

adding (1) and (2) we get 2I = $\int_0^{2\pi} \frac{2\pi\sin^2 nx}{\sin^{2n}x+\cos^{2n}x} dx = 2\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$

$= 2\pi \int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ (By prop.) $\left(\therefore \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx\right)$

$I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ ....(3)

$= 4\pi \int_0^{\pi/2} \frac{\sin^{2n}\left(\frac{\pi}{2}-x\right)}{\sin^{2n}\left(\frac{\pi}{2}-x\right)+\cos^{2n}\left(\frac{\pi}{2}-x\right)} dx$ (By prop.)

$I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n}x}{(\cos^{2n}x+\sin^{2n}x)} dx$ ....(4)

adding (3) and (4) we get $2\,I = 4\pi \int_0^{\pi/2} 1.dx$. Hence I = π^2.

Ex.9 Evalute $\int_0^1 \frac{dx}{(5+2x-2x^2)(2+e^{2-4x})}$

Sol. Let I = $\int_0^1 \frac{dx}{(5+2x-2x^2)(2+e^{2-4x})}$ ....(1)

$= \int_0^1 \frac{dx}{[5+2(1-x)-2(1-x)^2][1+e^{2-4[1-x]}]}$ (By Property)

$= \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{-2+4x})} = \int_0^1 \frac{e^{(2-4x)}}{(5+2x-2x^2)(1+e^{2-4x})} dx$ ....(2)

Adding (1) and (2) we get 2I = $\int_0^1 \frac{(1+e)^{(2-4x)}dx}{(5+2x-2x^2)(1+e^{(2-4x)})} = \int_0^1 \frac{dx}{(5+2x-2x^2)} = -\frac{1}{2}\int_0^1 \frac{dx}{\left(x^2-x-\frac{5}{2}\right)}$

$= -\frac{1}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{11}}{2}\right)^2} = \frac{1}{2}\int_0^1 \frac{dx}{\left(\frac{\sqrt{11}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \frac{1}{2}.\frac{1}{2.\frac{\sqrt{11}}{2}}\left[\ell n\left|\frac{\frac{\sqrt{11}}{2}+x-\frac{1}{2}}{\frac{\sqrt{11}}{2}-x+\frac{1}{2}}\right|\right]_0^1$

$$= \frac{1}{2\sqrt{11}}\ell n\left|\frac{1+\sqrt{11}}{\sqrt{11}-1}\right| - \frac{1}{2\sqrt{11}}\ell n\left|\frac{\sqrt{11}-1}{\sqrt{11}+1}\right| \qquad = \frac{1}{2\sqrt{11}}\ell n\left|\frac{1+\sqrt{11}}{\sqrt{11}-1}\right| + \frac{1}{2\sqrt{11}}\ell n\left|\frac{\sqrt{11}+1}{\sqrt{11}-1}\right|$$

$$= \frac{2}{2\sqrt{11}}\ell n\left|\frac{\sqrt{11}+1}{\sqrt{11}-1}\right| = \frac{1}{2\sqrt{11}}\ell n\left|\frac{\sqrt{11}+1}{\sqrt{11}-1}\right| \qquad \text{Hence } \ell = \frac{1}{4\sqrt{11}}\ \ell n\left|\frac{\sqrt{11}+1}{\sqrt{11}-1}\right|$$

Ex.10 Evaluate $\int_{-1}^{15} \text{Sgn}(\{x\})\,dx$, (where { * } denotes the fractional part function)

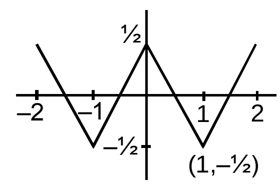
Sol. We have Sgn ({x}) = $\begin{cases} 1, \text{If } x \text{ is not an integer} \\ 0, \text{If } x \text{ is an integer} \end{cases}$ $\quad\therefore\quad \int_{-1}^{15} \text{Sgn}(\{x\})\,dx = \int_{-1}^{0} \text{Sgn}(\{x\})\,dx + \int_{0}^{15} \text{Sgn}(\{x\})\,dx$

$= \int_{-1}^{0} 1.\,dx + 15\int_{0}^{1} 1.\,dx$ = 1(0 + 1) + 15(1 – 0) = 16 $\qquad$ ($\because$ {x} is a periodic function)

Ex.11 Let the function f be defined by f(x) = |x – 1| – $\frac{1}{2}$, $0 \le x \le 2$, f(x + 2) = f(x) for all $x \in R$.

Evaluate **(i)** $\int_{0}^{100} f(x)\,dx$ $\qquad$ **(ii)** $\int_{0}^{1} |f(2x)|\,dx$

Sol. $\int_{0}^{2} f(x)\,dx = 0$ from the property of periodic function $\int_{0}^{100} f(x)\,dx = 50\int_{0}^{2} f(x)\,dx = 0$



Ex.12 Evaluate $\dfrac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx}$, where [x] and {x} denotes the integral part, and fractional part function of x and $n \in N$.

Sol. Let I = $\dfrac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx} = \dfrac{\int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + ... + \int_{n-1}^n [x]\,dx}{n\int_0^4 \{x\}\,dx}$

$$= \frac{0 + 1.\int_1^2 dx + 2\int_2^3 dx + ... + (n-1)\int_{n-1}^{n} dx}{n.\int_0^4 x\,dx} = \frac{0+1+2+3+......+(n-1)}{n.\frac{1}{2}} = \frac{\frac{(n-1)n}{2}}{\frac{n}{2}} = (n-1).$$

Ex.13 Show that $\int_0^{p+q\pi} |\cos x|\,dx$ = 2q + sin p where $q \in N$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$.

Sol. Let I = $\int_0^{p+q\pi} |\cos x|\,dx = \int_0^{q\pi} |\cos x|\,dx + \int_0^{p+q\pi} |\cos x|\,dx = q\int_0^{\pi} |\cos x|\,dx + \int_0^{p} |\cos x|\,dx$

{$\because$ period of |cos x| is π}

$$= q\left\{\int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx\right\} = \int_0^{p} \cos x\,dx = q\left\{\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx\right\} + \int_0^{p} \cos x\,dx$$

$$= q\,\{(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\} + (\sin x)_0^{p} = q\{(1-0) - (0-1)\} + \sin p - \sin 0 = 2q + \sin p$$

Ex.14 Evaluate $\int_0^{2n\pi}[\sin x+\cos x]\,dx$, (where [∗] is the greatest integer function)

Sol. Let I = $\int_0^{2n\pi}[\sin x+\cos x]$, [P] = $\begin{cases} 1 \ , & 0\le x<\dfrac{\pi}{2} \\ 0 \ , & \dfrac{\pi}{2}\le x<\dfrac{3\pi}{4} \\ -1 \ , & \dfrac{3\pi}{4}\le x<\pi \\ -2 \ , & \pi\le x<\dfrac{3\pi}{2} \\ -1 \ , & \dfrac{3\pi}{2}\le x<\dfrac{7\pi}{4} \\ 0 \ , & \dfrac{7\pi}{4}\le x<2\pi \end{cases}$

So, $\int_0^{2\pi}[\sin x+\cos x]dx=\int_0^{\pi/2}1.dx+\int_{\pi/2}^{3\pi/4}0.dx+\int_{3\pi/4}^{\pi}(-1)dx+\int_{\pi}^{3\pi/2}(-2)dx+\int_{3\pi/2}^{7\pi/4}(-1)dx+\int_{7\pi/4}^{2\pi}0.dx$

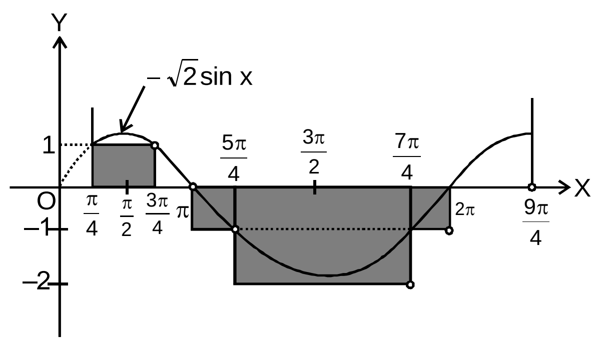
$= \dfrac{\pi}{2} + 0 - \pi + \dfrac{3\pi}{4} - 3\pi + 2\pi - \dfrac{7\pi}{4} + \dfrac{3\pi}{2} + 0 = -\pi$

Since sin x + cos x is periodic function with period 2π, so

I = $\int_0^{2n\pi}[\sin x+\cos x]\,dx=n\int_0^{2\pi}[\sin x+\cos x]\,dx = -n\pi$

Alternative Method : (Graphical Method)

It is clear from the figure.



$\int_0^{2n\pi}\left[\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)\right]dx$

$= \int_{\pi/4}^{9\pi/4}[\sqrt{2}\sin x]\,dx =$ Area of Shaded region

$= \left(\dfrac{3\pi}{4}-\dfrac{\pi}{4}\right) \times 1 + 10 + \left(\dfrac{5\pi}{4}-\pi\right) \times -1 + \left(\dfrac{7\pi}{4}-\dfrac{5\pi}{4}\right) \times -2 + \left(2\pi-\dfrac{7\pi}{4}\right) \times -1 = -\pi$

Hence $\int_0^{2\pi x}[\sin x+\cos x]\,dx=-n\pi$

C. DERIVATIVE OF ANTIDERIVATIVE (LEIBNITZ RULE)

If h(x) & g(x) are differentiable function of x then, $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$

Ex.15 Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Sol. Since $f(t) = \sqrt{1+t^2}$ is continuous, therefore $g'(x) = \sqrt{1+x^2}$

Ex.16 If $F(t) = \int_0^t \frac{1}{x^2+1}\,dx$, find F'(1), F'(2), and F'(x).

Sol. The integrand in this example is the continuous function f defined by $f(x) = \frac{1}{x^2+1}$.

$F'(t) = f(t) = \frac{1}{t^2+1}$. In particular, $F'(1) = \frac{1}{1^2+1} = \frac{1}{2}$, $F'(2) = \frac{1}{2^2+1} = \frac{1}{5}$,

Ex.17 Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$.

Sol. Let $u = x^4$. Then $\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{dx}\int_1^{u} \sec t\,dt = \frac{d}{du}\left(\int_1^{u} \sec t\,dt\right)\frac{du}{dx} = \sec u\,\frac{du}{dx} = \sec(x^4)\cdot 4x^3$.

Ex.18 FInd the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\,dt$

Sol. $F'(x) = \frac{dF}{du}\frac{du}{dx} = \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} = (\cos u)(3x^2) = (\cos x^3)(3x^2)$

$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$ $\qquad F'(x) = (\cos x^3)(3x^2)$.

Ex.19 Let $f(x) = \int_0^x \{(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)\}dt$. Find the value of 'a' for which f'(x) = 0 has two distinct real roots.

Sol. Differentiating the given equation, we get $f'(x) = (a-1)(x^2+x+1)^2 - (a+1)(x^2+x+1)(x^2-x+1)$.
Now, $f'(x) = 0 \Rightarrow (a-1)(x^2+x+1) - (a+1)(x^2-x+1) = 0 \Rightarrow x^2 - ax + 1 = 0$.
For distinct real roots D > 0 i.e. $a^2 - 4 > 0 \Rightarrow a^2 > 4 \Rightarrow a \in (-\infty, -2) \cup (2, \infty)$

Ex.20 Show that for a differentiable function f(x), $\int_0^n f'(x)\left\{[x] - x + \frac{1}{2}\right\}dx = \int_0^n f(x)\,dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r)$,

(where [∗] denotes the greaetest integer function and $n \in N$)

Sol. $$I = \int_0^n f'(x)[x]\,dx - \int_0^n x f'(x)\,dx + \frac{1}{2}\int_0^n f'(x)\,dx \;=\; \sum_{r=1}^{n}\int_{r=1}^{r} f'(x)[x]\,dx - \left\{(xf(x))_0^n - \int_0^n f(x)\,dx\right\} + \frac{1}{2}(f(x))_0^n$$

$$= \sum_{r=1}^{n}(r-1)\int_{r=1}^{r} f'(x)\,dx - nf(n) + \frac{1}{2}f(n) - \frac{1}{2}f(0) + \int_0^n f(x)\,dx \;=\; \sum_{r=1}^{n}(r-1)\{f(r)-f(r-1)\} - nf(n) + \frac{1}{2}f(n) + \int_0^n f(x)\,dx - \frac{1}{2}f(0)$$

$$= -f(1) - f(2) - \ldots\ldots\ldots - f(n-1) - f(n) + \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)\,dx \;=\; \sum_{r=1}^{n} f(r) + \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)\,dx$$

Ex.21 Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

Sol. $$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$;

$$\int_t^0 xe^x\,dx = xe^x\Big|_t^0 - \int_t^0 e^x\,dx = -te^t - 1 + e^t$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hopital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}(-e^t) = 0$$

Therefore $\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}(-te^t - 1 + e^t) = -0 - 1 + 0 = -1$

Ex.22 Evaluate $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$.

Sol. $$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx$$

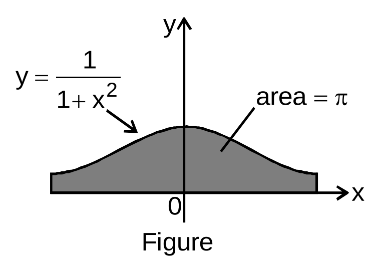
We must now evaluate the integrals on the right side separately :

$$\int_0^{\infty}\frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2} = \lim_{t\to\infty}\tan^{-1}x\Big]_0^t \quad = \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0\frac{dx}{1+x^2} = \lim_{t\to\infty}\tan^{-1}x\Big]_t^0 = \lim_{t\to\infty}(\tan^{-1}0 - \tan^{-1}t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Since $1/(1 + x^2) > 0$, the given improper integral can be interpreated as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x-axis (see Figure).



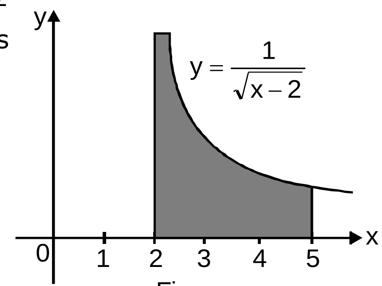
Figure

Ex.23 Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Sol. We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote x = 2. Since the infinite discontinuity occurs at the left end point of [2, 5]

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5 = \lim_{t\to 2^+}(2\sqrt{3} - \sqrt{x-2}) = 2\sqrt{3}$$

Thus, the given improper integrat is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure.



Figure

Ex.24 Evaluate $\int_0^1 \ln x\,dx$.

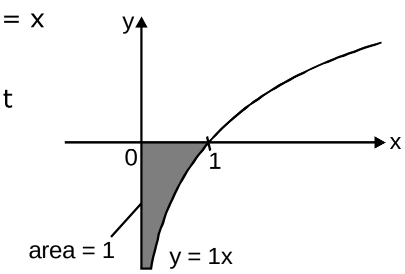
Sol. We know that the function f(x) = ln x has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ell n\, x = -\infty$. Thus, the given integral is improper and we have $\int_0^1 \ln x\,dx = \lim_{x\to 0^+}\int_t^1 \ln x\,dx$

Now we integrate by parts with u = ln x, dv = dx, du = dx/x, and v = x

$$\int_t^1 \ln x\,dx = x\ln x]_t^1 - \int_t^1 dx = 1\ln - t\ln t - (1-t) = -t\ln t - 1 + t$$

To find the limit of the first term we use I'Hopital's Rule :

$$\lim_{x\to 0^+} t\ln t = \lim_{x\to 0^+}\frac{\ln t}{1/t} = \lim_{t\to 0^+}\frac{1/t}{-1/t^2} = \lim_{t\to 0^+}(-t) = 0$$



Figure

Therefore $\int_0^1 \ln x\,dx = \lim_{t\to 0^+}(-t\ln t - 1 + t) = -0 - 1 + 0 = -1$

Figure shows the geometric interpretation of this result. The area of the shaded region above y = ln x and below the x-axis is 1.

Ex.25 Evaluate $\int_0^\infty [2e^{-x}]\,dx$, (where [∗] denotes the greatest integer function)

Sol. Let $I = \int_0^\infty [2e^{-x}]\,dx$. Let $y = 2e^{-x}$ ∴ $\frac{dy}{dx} = -2e^{-x} < 0 \ \forall\ x \in [0, \infty)$

∴ $2e^{-x}$ is decreaisng function $\forall\ x \in [0, \infty)$ $\Rightarrow$ $0 < 2e^{-x} \le 2\ \forall\ x \in [0, \infty)$

for $x > \ell n\, 2 \Rightarrow e^x > 2 \Rightarrow e^{-x} < \frac{1}{2} \Rightarrow 2e^{-x} < 1$ ∴ $0 \le 2e^{-x} < 1$ $\quad [2e^{-x}] = 0$

$$\therefore\quad I = \int_0^{\ell n 2}[2e^{-x}]\,dx + \int_{\ell n 2}^{\infty}[2e^{-x}]\,dx = \int_0^{\ell n 2} 1.\,dx + \int_{\ell n 2}^{\infty} 0.\,dx = (\ell n\, 2 - 0) + 0 = \ell n\, 2$$

D. DEFINITE INTEGRAL AS LIMIT OF A SUM

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{limit}}\ h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots\ldots + f(a + \overline{n-1}h)]$$

$$= \underset{h\to 0}{\text{limit}}\ h\sum_{r=0}^{n-1} f(a+rh) \quad \text{where } b - a = nh$$

If $a = 0$ & $b = 1$ then, $\underset{n\to\infty}{\text{limit}}\ h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$ **; where $nh = 1$**

or $\underset{n\to\infty}{\text{limit}}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

Remark : The symbol $\int$ was introduced by Leibnitz and is called integral sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, f(x) is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called integration.

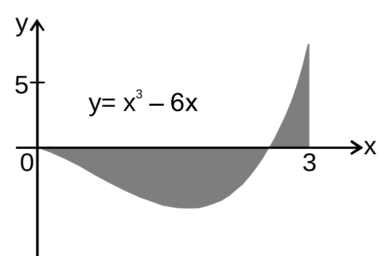
Ex.26 Evaluate $\int_0^3 (x^3 - 6x)\,dx$ using limit of sum.

Sol. $\int_0^3 (x^3 - 6x)\,dx = \lim\limits_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x - \lim\limits_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} = \lim\limits_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18}{n}i\right] = \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\} = \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$



Figure

$$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$$

This integral can't be interpreated as an area because f takes on both positive and negative values. But is can be interpreated as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure

E. ESTIMATE OF DEFINITE INTEGRATION & GENERAL INEQUALITY

STATEMENT : If f is continuous on the interval [a, b], there is atleast one number c between a and b such that $\int_a^b f(x)\,dx = f(c)\,(b-a)$

Proof : Suppose M and m are the largest and smallest values of f, respectively, on [a, b]. This means that $m \le f(x) \le M$ when $a \le x \le b$ $\Rightarrow$ $\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$ Dominance rule

$\Rightarrow$ $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ $\Rightarrow$ $m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$

Because f is continuous on the closed interval [a, b] and because the number $I = \frac{1}{b-a}\int_a^b f(x)\,dx$ lies between m and M, the intermediate value theorem syas there exists a number c between a and b for which f(c) = I ; that is, $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ $\int_a^b f(x)\,dx = f(c)\,(b-a)$

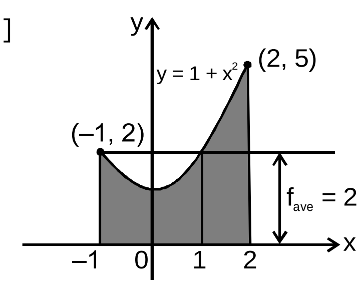
The mean value theorem for integrals does not specify how to determine c. It simply guarantees the existence of atleast one number c in the interval.

Since $f(x) = 1 + x^2$ is continuous on the interval [–1, 2], the Mean Value Theorem for Integrals says there is a number c in [–1, 2] such that $\int_{-1}^{2}(1+x^2)\,dx = f(c)\,[2-(-1)]$

In this particular case we can find c explicitly. From previous Example we know that $f_{ave} = 2$, so the value of c satisfies $f(c) = f_{ave} = 2$

Therefore $1 + c^2 = 2$ so $c^2 = 1$

Thus, in this case there happen to be two numbers c = ± 1 in the interval [–1, 2] that work in the mean value theorem for Integrals.



Figure

F. WALLI'S FORMULA & REDUCTION FORMULA

$$\int_0^{\pi/2} \sin^n x.\cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 3][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)...1 \text{ or } 2}K$$

Where K $= \frac{\pi}{2}$ **if both m and n are even (m, n $\in$ N) ;**

= 1 otherwise

Ex.27 Prove that, $\int \sin n\theta \sec\theta\,d\theta = -\frac{2\cos(n-1)\theta}{n-1} - \int \sin(n-2)\theta \sec\theta\,d\theta$.

Hence or otherwise evaluate $\int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos\theta}\,d\theta$.

Sol. Consider $\sin n\theta + \sin(n-2)\theta = 2\sin(n-1)\theta\cos\theta \Rightarrow \sin n\theta\sec\theta = 2\sin(n-1)\theta - \sin(n-2)\theta\sec\theta$

Hence $\int \sin n\ \sec\theta\, d\theta = -\dfrac{2}{(n-1)}\cos(n-1)\theta - \int \sin(n-2)\ \sec\theta\, d\theta$

Now $\quad \dfrac{1}{2}\displaystyle\int_0^{\pi/2} \dfrac{2\sin 3\theta\cos 5\theta}{\cos\theta}\, d\theta = \dfrac{1}{2}\int_0^{\pi/2}\dfrac{\sin 8\theta - \sin 2\theta}{\cos\theta} \qquad I = \dfrac{1}{2} I_8 - 1$

$$I_8 = -\frac{2}{7}\cos 7\theta\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 6\theta}{\cos\theta}d\theta. = \frac{2}{7} - \left[-\frac{2}{5}\cos 5\theta\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 4\theta}{\cos\theta}d\theta\right] = \frac{2}{7} - \left[\frac{2}{5} - \left\{-\frac{2}{3}\cos 3\theta\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 2\theta}{\cos\theta}d\theta\right\}\right]$$

$$= \frac{2}{7} - \left[\frac{2}{5} - \frac{2}{3} + 2\right] = \frac{2}{7} - \frac{2}{5} + \frac{2}{3} - \frac{2}{1} = \frac{30-42+70-210}{105} = -\frac{152}{105}$$

$$I = -\frac{152}{2\times 105} - 1 = -\frac{76+105}{105} = -\frac{181}{105}$$

Ex.28 Prove that $\displaystyle\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}$

Sol. L.H.S. $= \displaystyle\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\, d\theta = \int_0^{\pi/4}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta$ (Put $\sqrt{2}\ \sin\theta = \sin t \Rightarrow \cos\theta\, d\theta = \dfrac{\cos t}{\sqrt{2}}\ dt$)

when $\theta \to 0$ then $t \to 0$; $\theta \to \pi/4$ then $t \to \pi/2$

$\therefore$ L.H.S. $= \displaystyle\int_0^{\pi/2}\frac{\cos^4 t}{\sqrt{2}}\, dt = \frac{1}{\sqrt{2}}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi}{16\sqrt{2}}$ = R.H.S. (From Walli's formula)

Ex.29 If $u_n = \displaystyle\int_0^{\pi/2}\frac{\sin^2 nx}{\sin^2 x}dx$, then show that u_1, u_2, u_3 ,...... constitute an arithmetic progression. Hence or otherwise find the value of u_n.

Sol. $u_{n+1} - 2u_n + u_{n-1} = (u_{n+1} - u_n) - (u_n - u_{n-1})$

$$= \int_0^{\pi/2}\frac{(\sin^2(n+1)x - \sin^2 nx) - (\sin^2 nx - \sin^2(n-1)x}{\sin^2 x}dx = \int_0^{\pi/2}\frac{(\sin(2n+1)x\sin x - \sin(2n-1)x\sin x)}{\sin^2 x}dx$$

$$= \int_0^{\pi/2}\frac{(\sin(2n+1)x - \sin(2n-1)x)}{\sin x} = \int_0^{\pi/2}\frac{2\cos 2nx\sin x}{\sin x} = 2\int_0^{\pi/2}\cos 2nx\, dx = 2\ .\ \frac{\sin 2nx}{2n}\Big|_0^{\pi/2}$$

$= \dfrac{1}{n}(\sin n\pi - \sin 0) = 0 - 0 = 0 \qquad \therefore \quad u_{n-1} + u_{n+1} = 2u_n \qquad$ i.e., u_{n-1}, u_n, u_{n+1} form an A.P.

$\Rightarrow \ u_1, u_2, u_3$,.........constitute an A.P.

Ex.30 Evaluate $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$.

Sol. Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\,dx = \int_0^1 \cot^{-1}(1 - x(1 - x))dx$ $\qquad (\because\ 0 \le x < 1)$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x(1-x)}\right)dx = \int_0^1 \tan^{-1}\left(\frac{x+(1+x)}{1-x(1-x)}\right)dx = \int_0^1 (\tan^{-1}x + \tan^{-1}(1-x))\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-(1-x))\,dx = 2\int_0^1 \tan^{-1}x\,dx$$

Integrating by parts taking unity as the second function, we have

$$I = 2\left[\left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right] = 2\left[\frac{\pi}{4} - \frac{1}{2}[\ln|1+x^2|]_0^1\right] = 2\left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right] \quad \text{Hence } I = \frac{\pi}{2} - \ln 2.$$

Ex.31 Show that $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{(a^2-1)}}$. Hence or otherwise evaluate $\int_0^\pi \frac{dx}{(\sqrt{5}-\cos x)^3}$.

Sol. Let $I = \int_0^\pi \frac{dx}{(a-\cos x)}$ ....(1) $\qquad = \int_0^\pi \frac{dx}{a-\cos(\pi-x)}$ (By Prop.) $\qquad = \int_0^\pi \frac{dx}{(a+\cos x)}$ ....(2)

adding (1) and (2) then $2I = \int_0^\pi \frac{2a\,dx}{(a^2-\cos^2 x)} = 2a\,.\,2\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)}$

$$\Rightarrow \quad I = 2a\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)} \qquad = 2a\int_0^{\pi/2} \frac{\sec^2 dx}{a^2(1+\tan^2 x)-1} \qquad = 2a\int_0^{\pi/2} \frac{\sec^2 x\,dx}{(a^2-1)+(a\tan x)^2}$$

Put $a\tan x = t \quad \Rightarrow \quad a\sec^2 x\,dx = dt$ when $x = 0 \quad \Rightarrow \quad t = 0$; $x = \pi/2 \ \Rightarrow \quad t = \infty$

$$\text{then } I = 2\int_0^\infty \frac{dt}{(\sqrt{a^2-1})^2+t^2} = \frac{2}{\sqrt{(a^2-1)}}\left\{\tan^{-1}\left(\frac{t}{\sqrt{(a^2-1)}}\right)\right\}_0^\infty = \frac{2}{\sqrt{(a^2-1)}}\{\tan^{-1}\infty - \tan^{-1}0\} = \frac{2}{\sqrt{(a^2-1)}}\left\{\frac{\pi}{2}-0\right\}$$

Hence $I = \frac{\pi}{\sqrt{(a^2-1)}}$ or $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{(a^2-1)}}$

Differentiating both side w.r.t. 'a', we get $-\int_0^\pi \frac{dx}{(a-\cos x)^2} = \frac{-\pi a}{(a^2-1)^{3/2}}$

again differentiating both sides w.r.t. 'a' we get $2\int_0^\pi \frac{dx}{(a-\cos x)^3} = \frac{\pi(2a^2+1)}{(a^2-1)^{3/2}}$

Put $a = \sqrt{5}$ on both sides, we get $2\int_0^\pi \frac{dx}{(\sqrt{5}-\cos x)^3} = \frac{\pi(11)}{(4)^{3/2}}$ or $\int_0^\pi \frac{dx}{(\sqrt{5}-\cos x)^{3/2}} = \frac{11\pi}{16}$

Ex.32 Let f be an injective functions such that f(x) f(y) + 2 = f(x) + f(y) + f(xy) for all non negative real x and y with f(0) = 1 and f '(1) = 2 find f(x) and show that $\int f(x)\,dx - x\,(f(x) + 2)$ is a constant.

Sol. We have f(x) f(y) + 2 = f(x) + f(y) + f(xy) ......(1)

Putting x = 1 and y = 1 then f(1) f(1) + 2 = 3f(1)

we get f(1) = 1, 2 $\quad f(1) \neq 1 \quad (\because f(0) = 1$ & function is injective) then f(1) = 2

Replacing y by $\frac{1}{x}$ in (1) then $f(x)\, f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1) \Rightarrow f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \; (\because \; f(1) = 2)$

Hence f(x) is of the type $f(x) = 1 \pm x^n \Rightarrow f(1) = 1 \pm 1 = 2$ (given)

$\therefore \quad f(x) = 1 + x^n$ and $f'(x) = nx^{n-1} \quad \Rightarrow \quad f'(1) = n = 2 \quad \therefore \quad f(x) = 1 + x^2$

$\therefore \quad 3\int f(x)\,dx - x(f(x) + 2) = 3\int (1 + x^2)dx - x(1 + x^2 + 2) = 3\left(x + \frac{x^3}{3}\right) - x(3 + x^2) + c$

Ex.33 Evaluate $\int_0^{k\pi/2} (gof)\, x\, dx$, (If k is an even integer and g(x) = sin kx cot x and $f(x) = \frac{x}{k}$)

Sol. $\int_0^{k\pi/2} (gof)\, x\, dx = \int_0^{k\pi/2} g[(f(x)]\, dx = \int_0^{k\pi/2} g\left(\frac{x}{k}\right) dx = \int_0^{k\pi/2} \sin x \cot\left(\frac{x}{k}\right) dx = \int_0^{k\pi/2} \sin x \frac{(\cos x/k)}{(\sin x/k)}\, dx$

Let k = 2n

$$= \int_0^{n\pi} \frac{\sin x \cos\left(\frac{x}{2n}\right)}{\sin\left(\frac{x}{2n}\right)} dx = 2n\int_0^{\pi/2} \frac{\sin 2nt.\cot t}{\sin t} dt \qquad \left(\text{Put } \frac{x}{2n} = t \Rightarrow dx = 2n\, dt\right)$$

$$= 2n \int_0^{\pi/2} \cot t\, \{2\,(\cos t + \cos 3t + \text{.........} + \cos (2n - 1)\, t\}\, dt$$

$$= 2n \int_0^{\pi/2} [2\cos^2 t + 2\cos 3t \cos t + \text{..........} + 2\cos t\,(2n - 1)\, t]\, dt$$

$$= 2n \int_0^{\pi/2} [1 + \cos 2t + \cos 4t + \cos 2t + \text{.........} + \cos 2nt + \cos (2n - 2)\, t]\, dt$$

$$= 2n \int_0^{\pi/2} \left[t + \frac{\sin 2t}{2} + \frac{\sin 4t}{4} + \frac{\sin 2t}{2} + ... + \frac{\sin 2nt}{2n} + \frac{\sin(2n-2)t}{2n-2}\right]_0^{\pi/2} = n\pi.$$

Ex.34 Evaluate $\int_0^{\pi} \|\sin x| - |\cos x\|\, dx$

Sol. Let $I = \int_0^{\pi} \|\sin x| - |\cos x\|\, dx$ $\qquad$ Make |sin x| − |cos x| = 0 $\qquad \therefore \qquad$ |tan x| = 1

$\therefore \quad \tan x = \pm 1 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$ and both these values lie in the interval $[0, \pi]$.

We find for $0 < x < \frac{\pi}{4}$, |sin x| − |cos x| < 0 $\qquad \frac{\pi}{4} < x < \frac{3\pi}{4}$, |sin x| − |cos x| > 0

$\frac{3\pi}{4} < x < \pi$, |sin x| − |cos x| < 0

$\therefore \quad I = -\int_0^{\pi/4} (|\sin x| - |\cos x| \, dx + \int_{\pi/4}^{3\pi/4} |\sin x| - |\cos x|) \, dx - \int_{3\pi/4}^{\pi} (|\sin x| - |\cos x) \, dx$

$= -\int_0^{\pi/4} |\sin x| \, dx + \int_0^{\pi/4} |\cos x| \, dx + \int_{\pi/4}^{3\pi/4} |\sin x| dx - \int_{\pi/4}^{3\pi/4} |\cos x| dx - \int_{3\pi/4}^{\pi} |\sin x| dx + \int_{3\pi/4}^{\pi} |\cos x| dx$

$= -\int_0^{\pi/4} \sin x \, dx + \int_0^{\pi/4} \cos x \, dx + \int_{\pi/4}^{3\pi/4} \sin x \, dx - \int_{\pi/4}^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} \cos x \, dx - \int_{\pi/4}^{\pi} \sin x \, dx - \int_{3\pi/4}^{\pi} \cos x \, dx$

$= \left(\frac{1}{\sqrt{2}} - 1\right) + \left(\frac{1}{\sqrt{2}} - 0\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - 1\right) + \left(-1 + \frac{1}{\sqrt{2}}\right) - \left(0 - \frac{1}{\sqrt{2}}\right) = 4\sqrt{2} - 4$

Ex.35 Evaluate $\int_0^2 [x^2 - x + 1] \, dx$, (where [∗] is the greatest integer function)

Sol. Let $I = \int_0^2 [x^2 - x + 1] \, dx$

Let $f(x) = x^2 + x + 1 \Rightarrow f'(x) = 2x - 1$ for $x > 1/2$, $f'(x) > 0$ and $x < 12$, $f'(x)$, 0
Values of $f(x)$ at $x = 1/2$ and 2 are 3/4 and 3 integers between them an 1, 2 then $x^2 - x + 1 = 1, 2$

we get $x = 1$, $x = \frac{1+\sqrt{5}}{2}$ and values of $f(x)$ at $x = 0$ and 1/2 are 1 and 3/4 no integer between them

$\therefore \quad I = \int_0^{1/2} [x^2 - x + 1] \, dx + \int_{1/2}^{1} [x^2 - x + 1] \, dx + \int_1^{\frac{1+\sqrt{5}}{2}} [x^2 - x + 1] \, dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} [x^2 - x + 1] \, dx$

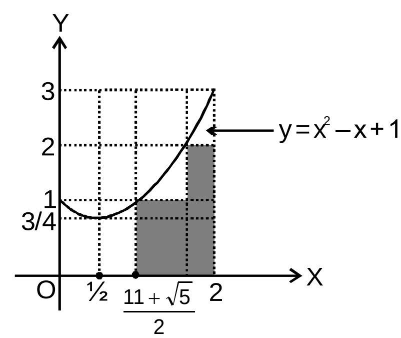
$= 0 + 0 + 1\int_1^{\frac{1+\sqrt{5}}{2}} 1.dx + 2\int_{\frac{1+\sqrt{5}}{2}}^{2} 1 \, dx \qquad = \left(\frac{1+\sqrt{5}}{2} - 1\right) + 2\left(2 - \frac{1+\sqrt{5}}{2}\right) = \left(\frac{5-\sqrt{5}}{2}\right)$

Alternative Method : It is clear from the figure

$\int_0^2 [x^2 - x + 1] \, dx$ = Area of bounded region

$= 0 + \left(\frac{1+\sqrt{5}}{2} - 1\right) \times 1 + \left(2 - \frac{1+\sqrt{5}}{2}\right) \times 2$

$= 3 - \left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{1-\sqrt{5}}{2}\right)$



Figure

Ex.36 Compute $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{an}\right)$ where a is a positive integer.

Calculate approximately $\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \ldots\ldots\ldots + \frac{1}{300}$

Sol. Let P = $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+......+\frac{1}{an}\right)$ ....(1)

$$=\lim_{n\to\infty}\left(\frac{1}{n+0}+\frac{1}{n+1}+\frac{1}{n+2}+......+\frac{1}{n+n(a-1)}\right)=\lim_{n\to\infty}\sum_{r=0}^{n(a-1)}\frac{1}{(n+r)}=\sum_{r=0}^{n(a-1)}\frac{1}{n(1+r/n)}=\int_0^{(a-1)}\frac{dx}{(1+x)}=[\ln(1+x)]_0^{n-1}$$

Hence P = ln a. Put n = 100, and a = 3 ln (1), we get $\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+........+\frac{1}{300}$ = ln 3

= 2.303 log 3 = (2.303) (0.4771213) = 1. 1 approx.

Ex.37 Evaluate $\int_1^{\infty}\frac{dx}{(x-\cos\alpha)\sqrt{(x^2-1)}}$, 0 < a < 2π

Sol. Let I = $\int_1^{\infty}\frac{dx}{(x-\cos\alpha)\sqrt{(x^2-1)}}$ (Put x – cos α = $\frac{1}{t}$ ⇒ dx = – $\frac{1}{t^2}$)

when x = 1 then t = $\frac{1}{1-\cos\alpha}$; x = ∞ then t = 0

$$\therefore\quad I=\int_{\frac{1}{1-\cos\alpha}}^{0}\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+\cos\alpha\right)^2-1}}=\int_0^{\frac{1}{1-\cos\alpha}}\frac{dt}{\sqrt{(1+t\cos\alpha)^2-t^2}}=\int_0^{\frac{1}{1-\cos\alpha}}\frac{dt}{\sqrt{(-t^2\sin^2\alpha+2t\cos\alpha+1)}}$$

$$=\frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}}\frac{dt}{\sqrt{-\left(t^2-\frac{2t\cos\alpha}{\sin^2\alpha}-\frac{1}{\sin^2\alpha}\right)}}=\frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}}\frac{dt}{\sqrt{-\left\{\left(t-\frac{\cos\alpha}{\sin^2\alpha}\right)^2-\frac{\cos^2\alpha}{\sin^4\alpha}-\frac{1}{\sin^2\alpha}\right\}}}$$

$$=\frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}}\frac{dt}{\sqrt{\left(\frac{1}{\sin^2\alpha}\right)^2-\left(t-\frac{\cos\alpha}{\sin^2\alpha}\right)^2}}=\frac{1}{|\sin\alpha|}\sin^{-1}\left(t\sin^2\alpha-\cos\alpha\right)\Big|_0^{\frac{1}{1-\cos\alpha}}$$

$$=\frac{1}{|\sin\alpha|}\left\{\sin^{-1}\left(\frac{\sin^2\alpha}{1-\cos\alpha}-\cos\alpha\right)-\sin^{-1}(0-\cos\alpha)\right\}=\frac{1}{|\sin\alpha|}\left\{\sin^{-1}(1)-\sin^{-1}(-\cos\alpha)\right\}$$

$$=\frac{1}{|\sin\alpha|}\left\{\frac{\pi}{2}-\sin^{-1}(-\cos\alpha)\right\}=\frac{\cos^{-1}(-\cos\alpha)}{|\sin\alpha|}=\frac{\cos^{-1}\cos(\pi-\alpha)}{|\sin\alpha|}=\frac{|\pi-\alpha|}{|\sin\alpha|}=\begin{cases}\frac{\pi-\alpha}{\sin\alpha}, & 0<\alpha<\pi\\ \frac{a-\pi}{-(\sin\alpha)}, & \pi<\alpha<2\pi\end{cases}.$$

Finally, I = $\frac{\pi-\alpha}{\sin\alpha}$

Ex.38 If $\int_0^{\pi}\left(\frac{x}{1+\sin x}\right)^2 dx = \lambda$ then show that $\int_0^{x}\frac{2x^2\cos^2(x/2)}{(1+\sin x)^2}dx = \lambda + 2\pi - \pi^2$.

Sol. Let $\int_0^{\pi}\frac{2x^2\cos^2(x/2)}{(1+\sin x)^2}dx = \int_0^{\pi}\frac{x^2(1+\cos x)}{(1+\sin x)^2}dx = \lambda + \int_0^{\pi} x^2\frac{\cos x}{(1+\sin x)^2}dx$

Integrating by parts taking x^2 as 1 st function, we we get $= \lambda + \left[x^2\left\{\frac{1}{(1+\sin x)}\right\}\right]_0^{\pi} + 2\int_0^{\pi}\left(\frac{x}{1+\sin x}\right)dx$ ...(1)

$I = \lambda - \pi^2 + 2\int_0^{\pi}\frac{x}{1+\sin x}dx$ [By Prop.] $= \lambda - \pi^2 + 2\int_0^{\pi}\frac{(\pi - x)dx}{1+\sin x}$ ....(2)

Adding (1) and (2) we get $2I = 2\lambda - 2\pi^2 + 2\pi\int_0^{\pi}\frac{x}{(1+\sin x)}$ or $I = \lambda - \pi^2 + \pi\int_0^{\pi}\frac{(1-\sin x)}{1-\sin^2 x}dx$

$= \lambda - \pi^2 + \pi\int_0^{\pi}(\sec^2 x - \sec x\tan x)dx$ $= \lambda - \pi^2 + \pi\{\tan x \_ \sec x\}_0^{\pi}$

$= \lambda - \pi^2 + \pi\{(0+1)-(0-1)\}$ $= \lambda - \pi^2 + 2\pi$ Hence $I = \lambda - 2\pi + \pi^2$

Ex.39 Prove that $\int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2 x + b^2\sin^2 x)}dx = \frac{\pi}{4ab^2(a+b)}$

Sol. Let $I = \int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2 x + b^2\sin^2 x)}dx$ Intergrating by parts taking x as a first function, we have

$$= \left[x\left\{\frac{-1}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}\right\}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}dx$$

$$= -\frac{\pi}{4(b^2-a^2)b^2} + \frac{1}{2(b^2-a^2)}\int_0^{\pi}\frac{\sec^2 x dx}{a^2 + (b\tan x)^2}$$ (Put b tanx = t $\Rightarrow$ $\sec^2 x\, dx = \frac{dt}{b}$)

$$= \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{2(b^2-a^2)}\int_0^{\infty}\frac{dt}{b(a^2+t^2)} = \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{2ab(b^2-a^2)}\left[\tan^{-1}\left(\frac{t}{a}\right)\right]_0^{\infty}$$

$$= \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{2ab(b^2-a^2)}\left(\frac{\pi}{2} - 0\right) = \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{4ab(b^2-a^2)} = \frac{\pi(b-a)}{4ab^2(b^2-a^2)}$$

Ex.40 Evaluate $\int_{-3/2}^{2}$ f(x) dx, where f(x) is given by f(x) = $\max_{-3/2\le t\le x}$ (|t – 1| – |t| + t +1)

Sol. Let g(t) = |t – 1| – |t| + |t + 1| $= \begin{cases} -t, & t<-1 \\ t+2, & -1<t<0 \\ 2-t, & 0<t<t \\ t, & t>1 \end{cases}$ Hence $\begin{cases} 3/2, & -3/2\le x<-1/2 \\ 2+x, & -1/2<x\le 0 \\ 2, & 0<x\le 2 \end{cases}$

$\therefore \int_{-3/2}^{2} f(x) = \int_{-3/2}^{-1/2} 3/2\,dx + \int_{-1/2}^{0} (2+x)\,dx + \int_{0}^{2} 2dx = \frac{3}{2}\left(-\frac{1}{2}+\frac{3}{2}\right)+0\left(-1+\frac{1}{8}\right) + 2\,(2-0) = \frac{51}{8}$

Ex.41 If $I_n = \int_{-\infty}^{0} e^x \sin^n x dx \quad \forall n \ge 2 \in N$, then prove that I_{n-2}, I_n, I_{n+2} cannot be G. P.

Sol. $I_n = \int_{-\infty}^{0} e^x \sin^n x dx = \left[\sin^n x \int e^x dx\right]_{-\infty}^{0} - \int_{-\infty}^{0} n \sin^{n-1} x \cos x\; e^x dx = -n \int_{-\infty}^{0} \sin^{n-1} x \cos x\; e^x dx$

$\int_{-\infty}^{0} \sin^{n-1} x \cos x\; e^x dx$

$= -n\left[\left[\sin^{n-1} x \cos x e^x\right]_{-\infty}^{0} - \int_{-\infty}^{0}\left((n-1)\sin^{n-2} x\cos^2 x - \sin^{n-1} x \sin x\right)e^x dx\right]$

$= n(n-1) \int_{-\infty}^{0}\left(\sin^{n-2} x\left(1-\sin^{2x}\right)\right)e^x dx - n\int_{-\infty}^{0} \sin^n x\, e^x dx - n$

$= n(n-1) \int_{-\infty}^{0} \sin^{n-2} x\, e^x dx - n(n-1)\int_{-\infty}^{0} \sin^n x\, e^x dx - n\int_{-\infty}^{0} \sin^n x\, e^x dx$

$\Rightarrow I_n = n(n-1)I_{n-2} - n(n-)I_n - nI_n \Rightarrow I_n(1+n^2) = n(n-1)\, I_{n-2}$

$\Rightarrow I_n = \frac{(n+1)(n+2)}{n^2+4n+5} I_n$ ....(1)

Now $I_{n+2} = \frac{(n+1)(n+2)}{n^2+4n+5} I_n$ ....(2)

From equation (1) and (2) $\frac{I_n}{I_{n-2}} = \frac{n(n-1)}{n^2+1}$ and $\frac{I_{n+2}}{I_n} = \frac{(n+1)(n+2)}{n^2+4n+5}$

Let I_{n-2}, I_n and I_{n+2} are in G. P., then

$\frac{I_n}{I_{n-2}} = \frac{I_{n+2}}{I_n} \Rightarrow \frac{n(n-1)}{n^2+1} = \frac{(n+1)(n+2)}{n^2+4n+5} \Rightarrow 2n^2 + 8n + 2 = 0$

which is not possible $\forall n \in N$.

$\Rightarrow$ I_{n-2}, I_n and I_{n+2} can't be in G. P.

Ex.42 For all positive integer k, prove that $\frac{\sin 2kx}{\sin x}$ = 2[cosx + cos 3x + ... + cos(2k – 1)x]

Hence prove that $\int_0^{\pi/2}$ sin 2kx cot x dx = $\pi/2$

Sol. We have 2 sinx [cos x + cos 3x + ...+ cos (2k–1)x]

= 2 sinx cosx + 2 sinx cos 3x + ...+ 2 sinx cos(2k – 1)x

= sin 2x + sin 4x – sin 2x + sin 6x – sin 4x + ... + sin 2kx – sin (2k – 2) x

= sin 2kx

$\Rightarrow$ 2[cos x + cos 3x + ... + cos (2k – 1) x] = $\frac{\sin 2kx}{\sin x}$

$$= 2\int_0^{\pi/2} \sin 2kx \cot x\, dx = \int_0^{\pi/2} \left(\frac{\sin 2kx}{\sin x}\right)\cos x\, dx$$

$$= \int_0^{\pi/2} [2\cos^2 x + 2\cos 3x \cos x + \ldots\ldots + 2\cos(2k-1)x\cos x]\, dx$$

$$= \int_0^{\pi/2} [\cos x + \cos 3x + \ldots\ldots + \cos(2k-1)x]\cos x\, dx$$

$$= \left[x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \ldots\ldots + \frac{\sin 2kx}{2k} + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} = \frac{\pi}{2}$$

Ex.43 Let f(x) is periodic function such that

$$\int_0^x (f(t))^3\, dt = \frac{1}{x^2}\left(\int_0^x (f(t)dt)\right)^3 \quad \forall x \in R - \{0\}$$

Find the function f(x) if (1) = 1 .

Sol. Let $\int_0^x (f(t))^3 dt$ = F(x)

$\Rightarrow$ f(x) = F'(x) ....(1)

$\therefore \int_0^x (f(t))^3 dt = \int_0^x \{F'(t)\}^3 dt$ ....(2)

and $\frac{1}{x^2}\left(\int_0^x (f(t)dt)\right)^3 = \frac{(F(x))^3}{x^2}$ ....(3)

from (2) and (3) $\int_0^x (F'(t))^3 dt = \frac{1}{x^2}\int_0^x (F(x))^3$

Differentiating bot sides w.r.t.x, we get

$$F'(x))^3 = \frac{x^2.3(F(x))^2F'(x) - (F(x))^3.2x}{x^4} = \frac{3(F(x))^2F'(x) - (F(x))^3}{x^3}$$

or $(x F'(x))^3 = 2x(F(x))^2F'(s) - 2(F(x))^3$

or $\left\{\frac{xF'(x)}{F(x)}\right\}^3 = 3\left\{\frac{xF'(x)}{F(x)}\right\} - 2$

$\Rightarrow$ $\lambda^3 - 3\lambda + 2 = 0$ where $\lambda = \frac{xF'(x)}{F(x)}$

$\Rightarrow$ $(\lambda-1)^2(\lambda+2) = 0$ $\therefore$ $\lambda = 1,-2,$

for $\lambda = 1$ $\frac{xF'(x)}{F(x)} = 1$

$\Rightarrow$ $\frac{F'(x)}{F(x)} = \frac{1}{x}$ $\therefore$ $\ln F(x) = \ln x + \ln c$

$\Rightarrow$ $F(x) = cx$ $\therefore$ $\ln F'(x) = c$

$f(x) = c$ {from (1)}

$f(1) = 1 = c$ ($\because$ $f(1) = 1$)

$f(x) = 1$ ......(4)

for $\lambda = -2$: $\frac{xF'(x)}{F(x)} = -2$

$\Rightarrow$ $\frac{xF'(x)}{F(x)} - \frac{2}{x}$ $\therefore$ $\ln F(x) = -2\ln x + \ln c_1$

$\Rightarrow$ $F(x) = \frac{c_1}{x^2}$ $\therefore$ $\ln F'(x) = -\frac{2c_1}{x^3}$

$\Rightarrow$ $f(x) = -\frac{2c_1}{x^3}$ $\Rightarrow$ $f(1) = 1 = -2c_1$

then $f(x) = \frac{1}{x^3}$

But given f(x) is a periodic function

Hence $f(x) = 1$

Ex.44 Let P_n denote the polynomial of degree n given by $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\ldots + \frac{x^n}{n} = \sum_{k=1}^{n} \frac{x^k}{k}$.

Then, for ever $x < 1$ and ever $n \geq 1$, prove that $-\log(1-x) = P_n(x) + \int_0^K \frac{u^n}{1-u}du$. ....(1)

Sol. From the algebraic identity $1 - u^n = (1-u)(1 + u + u^2 + \dots\dots\dots + u^{n-1})$,

we obtain the formula $\frac{1}{1-u} = 1 + u + u^2 + \dots.u^{n-1} + \frac{u^n}{1-u}$,

Which is valid for $u \neq 1$. Integrating this from 0 to x, where x < 1, we obtain (i)
We can rewrite (i) in the form
$-\log(1-x) = P_n(x) + E_n(x)$, ...(ii)

where $E_n(x)$ is the integral, $E_n(x) = \int_0^K \frac{u^n}{1-u}$.

Ex.45 Assume $\int_0^{\pi} \ln \sin\theta d\theta = -\pi \ln 2$ then prove that

$$\int_0^{\pi} \theta^3 \ln \sin\theta\, d\theta = \frac{3\pi}{2}\int_0^{\pi} \theta^2 \ln(\sqrt{2}\sin\theta)\, d\theta.$$

Sol. Let $I = \int_0^{\pi} \theta^3 \ln \sin\theta\, d\theta$ ....(1)

$= \int_0^{\pi} (\pi-\theta)^3 \ln \sin\theta\, d\theta$. [By Prop.]

$= \int_0^{\pi} (\pi^3 - 3\pi^2\theta + 3\pi\theta^2 - \theta^3) \ln \sin\theta\, d\theta$

$= \pi^3 \int_0^{\pi} \theta^3 \ln \sin\theta\, d\theta - 3\pi^2 \int_0^{\pi} \theta \ln \sin\theta\, d\theta + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta\, d\theta - \int_0^{\pi} \theta^3 \ln \sin\theta\, d\theta$

$= \pi^3 \int_0^{\pi} \theta^3 \ln \sin\theta\, d\theta - 3\pi^2 \int_0^{\pi} \theta \ln \sin\theta\, d\theta + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta\, d\theta - I$ [From (1)]

$\therefore 2I = \pi^3 I_1 - 3\pi^1 I_2 + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta\, d\theta$

Now $I_1 = \int_0^{\pi} \ln \sin d\theta = -\pi \ln 2$ (given)

$I_2 = \int_0^{\pi} \theta \ln \sin\theta\, d\theta = \int_0^{\pi} (\pi-\theta) \ln \sin(\pi-\theta)\, d\theta = \int_0^{\pi} (\pi-\theta) \ln \sin\theta\, d\theta$

$\therefore \quad 2I_2 = \pi \int_0^{\pi} \ln \sin\theta\, d\theta = -\pi^2 \ln 2$ (given)

$\therefore \quad I_2 = -\frac{\pi^2}{2} \ln 2$

then $2I = -\pi^2 \ln 2 \ \frac{3\pi^4}{2} \ \ln 2 + 3\pi\int_0^{\pi} \ \theta^2 \ln \sin\theta \, d\theta$

$\Rightarrow \quad I = \frac{\pi^4}{2}\ln 2 \ \ \frac{3\pi^4}{2} \ \int_0^{\pi} \ \theta^2 \ln \sin\theta \, d\theta$

$= \frac{3\pi}{2}\int_0^{\pi} \ \theta^2 \ln \sqrt{2} \, d\theta = \frac{3\pi}{2}\int_0^{\pi} \ \theta^2 \ln \sin q \, d\theta \qquad = \frac{3\pi}{2}\int_0^{\pi} \ \theta^2 \ln (\sqrt{2} \sin \theta) d\theta$

Ex.46 Evalute $\int_0^{\pi/2}$ cosesθ $\tan^{-1}$ (c sinθ)dθ.

Sol. Let $I = f(c) \Rightarrow f'(c) = \int_0^{\pi/2} \frac{\cos es\theta \sin\theta}{1 + c^2 \sin^2\theta} d\theta \Rightarrow f'(c) = \frac{\pi}{2\sqrt{c^2+1}}$

Now integrate to get $I = \frac{\pi}{2} \ln\left(c + \sqrt{c^2+1}\right)$

Ex.47 Use induction to prove that, $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx \ = \ \forall n \geq 2, \ n \in N$

Sol. $P(k) : \int_0^{\pi/2} \cos^{k-2} x \sin kx \, dx = \frac{1}{k-1}$

$P(k + 1) ; \int_0^{\pi/2} \cos^{k-1} x \sin(k+1) dx = \frac{1}{k} \qquad = \int_0^{\pi/2} \cos^{k-2} x \sin(k+1) x \cos x \, dx$

Now, We have sinkx = sin[(k + 1) x – x]

= sin(k + 1) x cos – cos(k + 1)x sin x

Hence sin (k + 1) x cosx = sinkx + cos(k + 1) x sin x

Subistuting in $P(k + 1) = \int_0^{\pi/2} \cos^{k-2}$ [sin kx + cos(k + 1) x sin x] dx

$P(k) + \int_0^{\pi/2} \cos^{k-2}$ sinx . cos(k + 1) x dx

Now I. B. P. to get the result